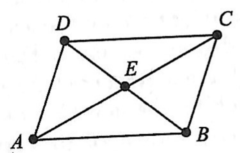
The New Standards

G-CO.3

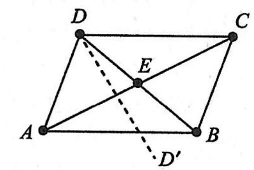
Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

A parallelogram is a quadrilateral that has both pairs of opposite sides parallel and also has the additional properties that: both pairs of opposite sides are congruent, pairs of opposite angles are congruent, diagonals bisect each other, and adjacent angles are supplementary. These features help determine how to use reflections or rotations to carry a parallelogram onto itself.

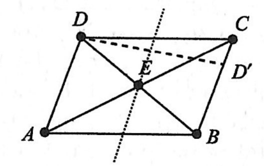


Since the diagonals of a parallelogram bisect each other, point E in the adjacent diagram of parallelogram $ABCD$ is the midpoint of $\overline{AC}$ and of $\overline{BD}$. It can be used as the center of a 180-degree rotation that maps $ABCD$ onto $CDAB$. Recognizing that there are no line reflections that map the parallelogram onto itself is not necessarily easy. For instance, one might think that it is possible to reflect the parallelogram onto itself over one of its diagonals. However, the diagonals of a parallelogram are not perpendicular to each other. So in order to reflect a point (or a set of points), it is necessary to map a point to its image by drawing a perpendicular line to the reflecting line and find the image point on that line on the other side of the reflecting line at the

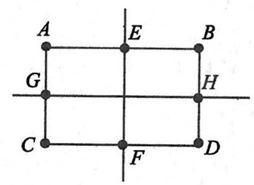
same distance from the reflecting line. In the diagram below, the image of point D would be D' over diagonal $\overline{AC}$ and not point B.



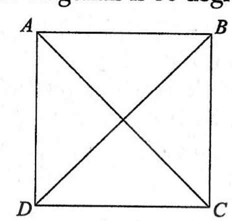
Another line to check as a reflecting line might be a line parallel to AD. Once again, a reflection moves a point over a line of reflection in a perpendicular manner. The diagram below shows the image of point D over a reflecting line that passes through point E is point D' and not the point C.



A rectangle has 180-degree rotation as described above for all parallelograms. However, it also has line reflections through any line that is a perpendicular bisector of any side of the rectangle.

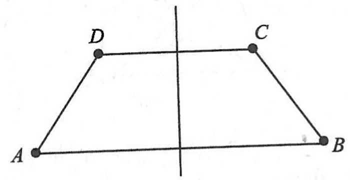


Remember that a square is a special type of rectangle that has perpendicular diagonals. Because of that, some rotations will map a square onto itself. The diagram below shows a square and its diagonals. Since the diagonals of a square are perpendicular to each other, a rotation through the center of the square (where the diagonals intersect) will map the square onto itself. Recall that the angle between these diagonals is 90 degrees.

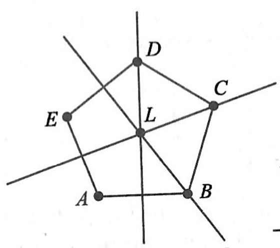


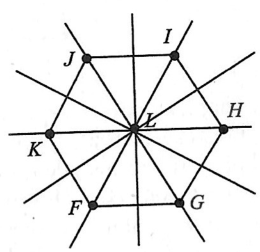
In fact in a square, there are rotations at any multiple of 90 degrees.

For a trapezoid, only an isosceles trapezoid will have a reflecting line that is the perpendicular bisector of its two bases that will map the figure onto itself. No rotations will accomplish this.



In a regular polygon with an even number of sides, the perpendicular bisector of any side will reflect the polygon to itself. Any diagonal of a regular polygon with an even number of sides will also reflect the polygon to itself. If a regular polygon has an odd number of sides, the perpendicular bisector of any side will pass through the opposite vertex. That line will be a line reflection that will reflect the polygon onto itself. However in all regular polygons, the place where the perpendicular bisectors meet is at a point inside the polygon that serves as the center of the polygon. Any integral multiple of angles formed by connecting this center to the endpoints of any one side can be used as a rotation centered on the center of the polygon will map the figure onto itself. Here are two examples.





A rotation that uses an angle that is a multiple of the measure of angle *BLC* will map the figure onto itself.

A rotation that uses an angle that is a multiple of the measure of angle *FLG* will map the figure onto itself.

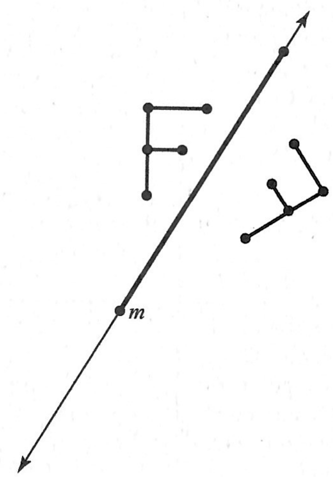
G-CO.5

Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g. (graph paper, tracing paper), or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

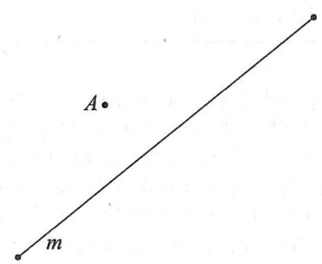
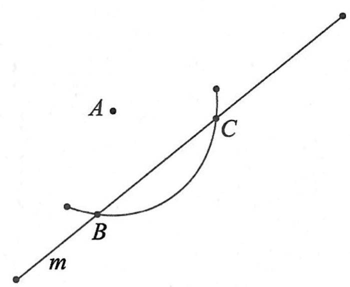
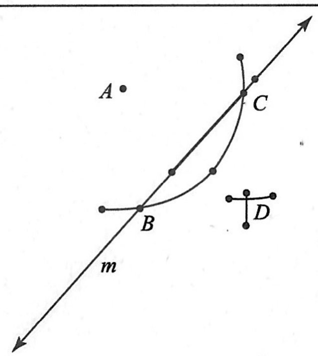
If point A is not on line m, the reflection image of A over line m is the point A' if and only if m is the perpendicular bisector of line segment $\overline{AA'}$. If point A is on line m, the reflection image of point A is A itself.

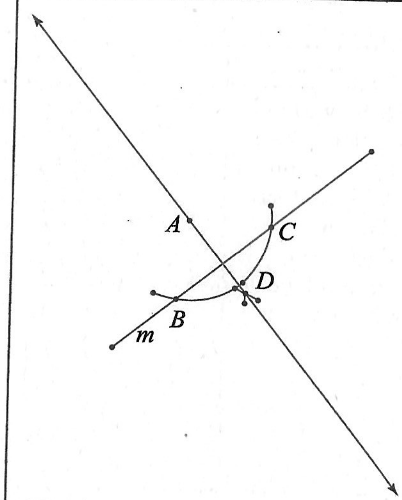
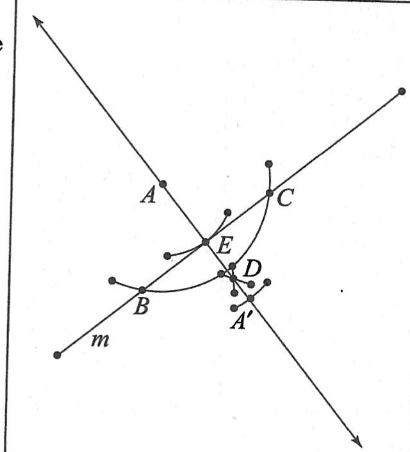
One way to draw a reflection image of a figure is to fold the paper on its reflecting line and trace the image on to the other side of the paper.

For instance, fold your paper over line m and trace the figure on the other side of the paper. See below.

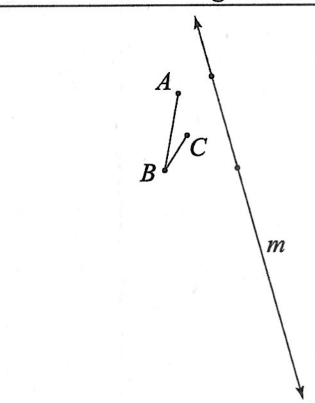
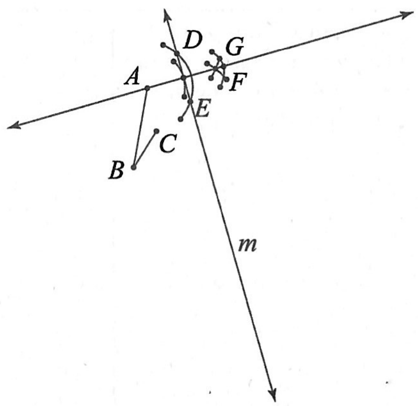


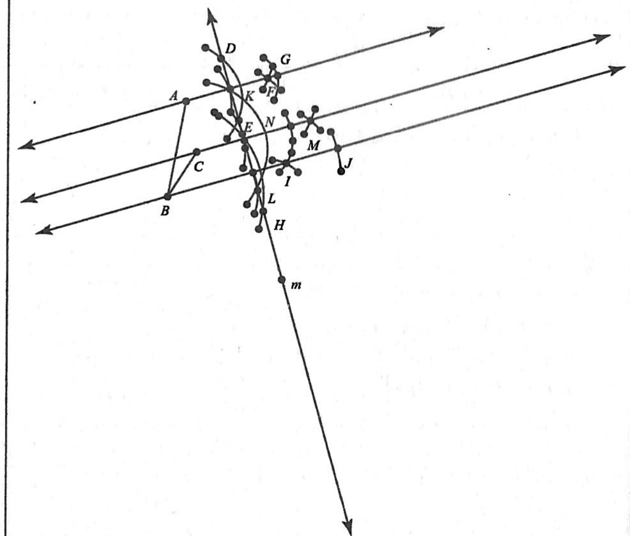
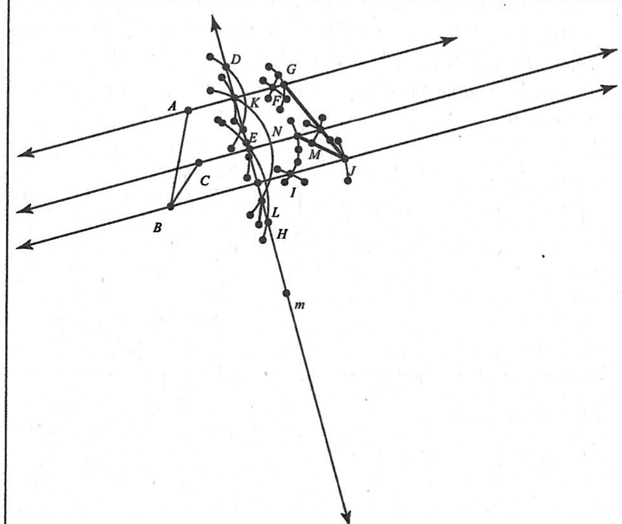
To construct the reflection image of a point over a reflecting line, simply use the fact that the reflecting line is a perpendicular bisector of a point and its image. Construct a line through the preimage point. Measure the distance from the preimage to the reflecting line on that perpendicular line, and measure the same distance on the other side of the perpendicular line.

Step	Diagram
Draw a line m and a point A that is not on line m.	
From point A, open your compass more than the distance from point A to line m. With the point of the compass at A, swing an arc that intersects line m in two places. Call them points B and C.	
Open your compass more than half the way from point B to point C. Swing arcs from these two points on the other side of line m from point A. Where these two arcs intersect, call that point D.	

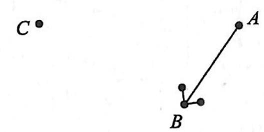
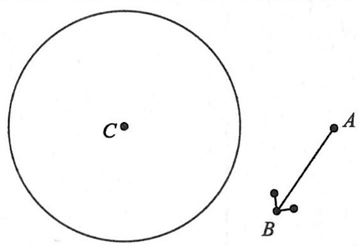
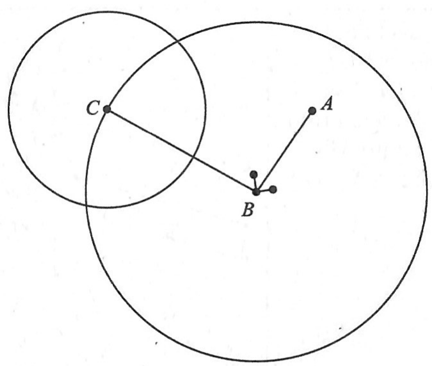
Step	Diagram
Draw line AD. Label the point where line AD intersects line m as point E.	
Measure the distance from point A to point E. Copy that distance on the other side of line AD over line m. Label the point found as A'.	

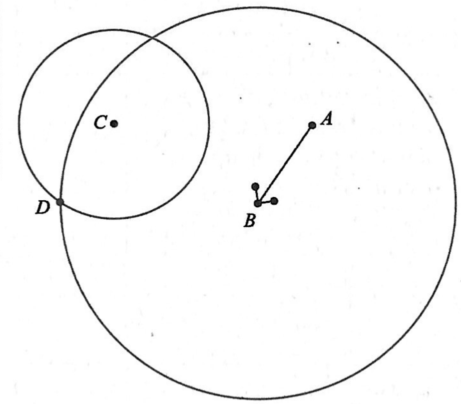
To construct the reflection image of a figure over a line, it is necessary to reflect each of the endpoints that define that figure.

Step	Diagram
Draw two line segments that share a common endpoint. Draw a line through which this figure will be reflected.	
From point A, construct its reflection image over line m. Call this point G.	

Step	Diagram
Repeat this process to find the reflection images of points B and C. Label the reflection image of point B as J and the reflection image of point C as N.	
Draw line segment $\overline{GJ}$ and $\overline{JN}$. The reflection image of figure ABC is figure GJN.	

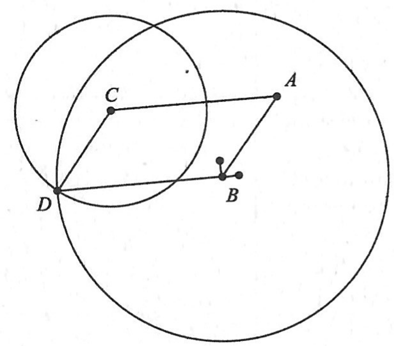
One way to translate a figure in a plane is to use a construction that slides the figure according to a vector. In order to understand the method used, let's just start with translating a point according to some vector.

Step	Diagram
1. Draw point C and vector $\overrightarrow{AB}$.	
2. Measure the length of vector $\overrightarrow{AB}$. Draw a circle from point C with a radius of length AB.	
3. Measure the distance from point A to point C. Use that as the radius of a circle centered at point B.	

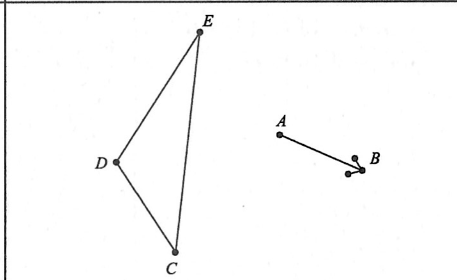
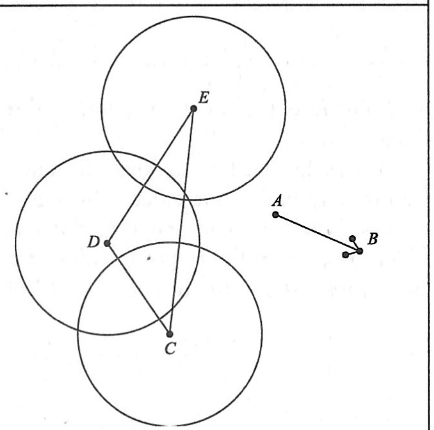
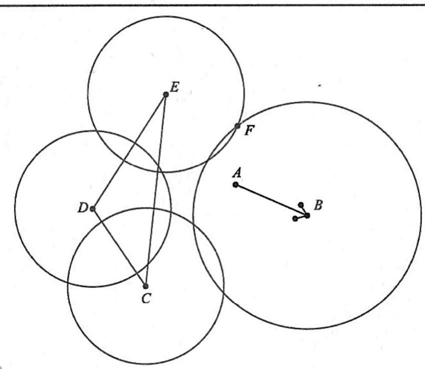
Step	Diagram
4. Label the point on circle C where circle B intersects it in the direction of vector $\overrightarrow{AB}$. Call that point D.	

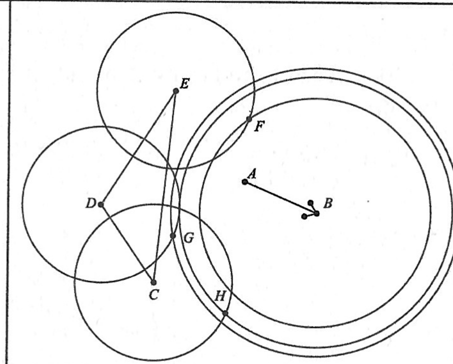
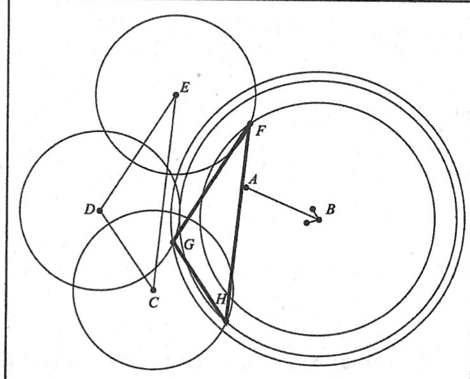
Point D is the image of point C through a translation in the direction of vector $\overrightarrow{AB}$.

It may be hard to understand why this construction works. The diagram below may make this clearer because it includes some additional line segments. Now one can see that the construction generated a parallelogram. Therefore, since the opposite sides of a parallelogram are congruent, point C is obviously slid in the direction of $\overrightarrow{AB}$ to point D.



Now it is possible to translate a geometric figure using this approach to the construction.

Step	Diagram
1. Start by drawing a figure, such as triangle $\triangle CDE$, and a vector, such as vector $\overrightarrow{AB}$.	
2. Measure the length of vector $\overrightarrow{AB}$. At points C, D, and E, construct circles with the length of $\overrightarrow{AB}$ as the radius and the respective points at the center.	
3. Measure the distance from point A to point E. Use that as the radius of a circle with center B. Label point F where this new circle intersects circle E in the same direction as vector $\overrightarrow{AB}$. (Note that point F is the translation of point E in the direction of vector $\overrightarrow{AB}$.)	

Step	Diagram
4. Repeat this same process to translate points C and D to points G and H.	
5. Draw triangle $\triangle FGH$, which is the translation image of $\triangle ECD$.	

A translation slides a point on the coordinate plane. Think of a translation as a function that maps a point on the plane to another point on the plane. For instance, a translation that moves a point to the right 3 units and upward 4 units can be described as:

$$T_{3,4}: (x, y) \rightarrow (x + 3, y + 4)$$

MATH FACTS

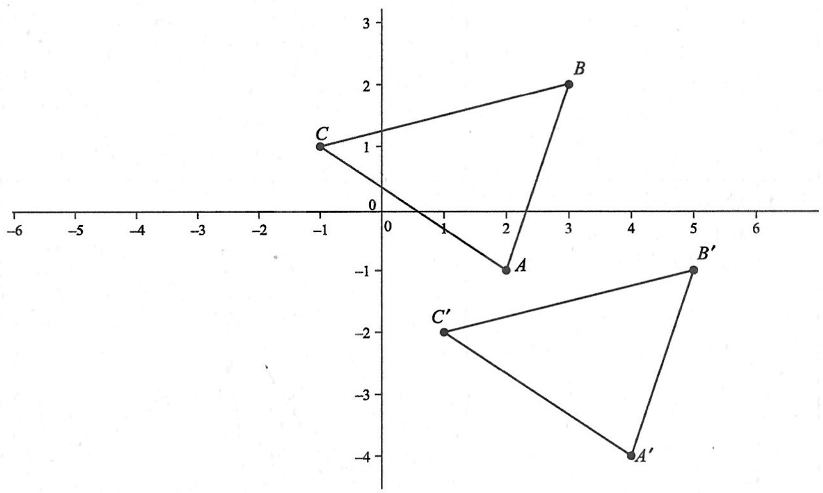
The translation $T_{a,b}$ is a mapping that moves the preimage to a point that is *a* units to the right and *b* units upward. If *a* is negative, the movement is to the left. If *b* is negative, the movement is downward.

Example 1

Given the vertices of $\triangle ABC$ are $A(2,-1)$, $B(3,2)$, and $C(-1,1)$, draw $T_{2,-3}(\triangle ABC)$.

Solution: The desired translation moves every point 2 units to the right and 3 units down.

$T_{2,-3}(2,-1) = (4,-4)$ $\quad T_{2,-3}(3,2) = (5,-1)$ $\quad T_{2,-3}(-1,1) = (1,-2)$



Therefore, the coordinates of the images of vertices of $\triangle ABC$ are $A'(4,-4)$, $B'(5,-1)$, and $C'(1,-2)$.

A rotation turns a figure around a point through a specific angle.

MATH FACTS

The rotation R_d is a mapping that moves the preimage to a point that rotates the point counterclockwise from the origin $d°$. Special formulas for rotations are listed below:

$$R_{90}: (x,y) \rightarrow (-y,x)$$

$$\text{A half turn or } R_{180}: (x,y) \rightarrow (-x,-y)$$

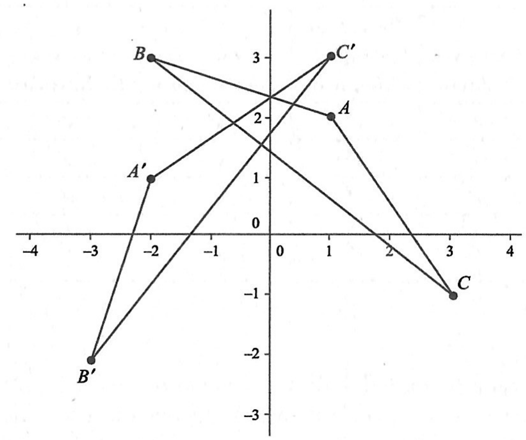
$$R_{270}: (x,y) \rightarrow (y,-x)$$

Example 2

Given the vertices of $\triangle ABC$ are $A(1,2)$, $B(-2,3)$, and $C(3,-1)$, draw $R_{90}(\triangle ABC)$.

Solution: The desired rotation moves every point counterclockwise from the origin at 90°.

$R_{90}(1,2) = (-2,1)$ $\quad R_{90}(-2,3) = (-3,-2)$ $\quad R_{90}(3,-1) = (1,3)$



Therefore, the image of the vertices of $\triangle ABC$ are $A'(-2,1)$, $B'(-3,-2)$, and $C'(1,3)$.

G-CO.6

Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure. When given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

There are several rigid motions, including a line reflection, a translation, a rotation, and a glide reflection. Two figures are congruent if, through a set of rigid motions, one figure can map to the other (or its image coincides with the other).

Rigid motions (or isometries) are all line reflections or compositions of line reflections. All of them preserve distance, angle measure, betweenness, and collinearity. However, if the rigid motion is composed of an odd number of reflections, the image has reversed orientation as compared with the preimage. If a rigid motion is composed of an even number of reflections, orientation is preserved.

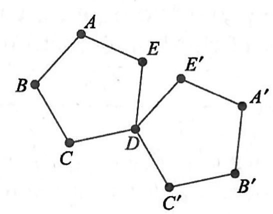
The chart below lists each of the basic rigid motions and the properties that each preserves.

Rigid Motion	Preserves Distance	Preserves Angle Measure	Preserves Betweenness	Preserves Collinearity	Preserves Orientation
Reflection	✔	✔	✔	✔	
Translation	✔	✔	✔	✔	✔
Rotation	✔	✔	✔	✔	✔
Glide reflection	✔	✔	✔	✔	

Example 1

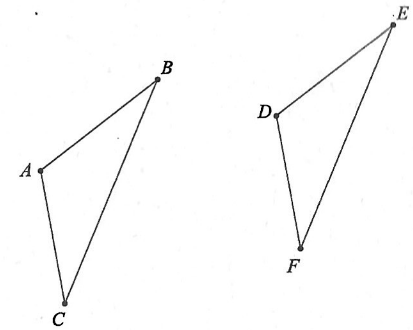
A regular pentagon is rotated 216° around one of its vertices. Identify which of the properties (distance, angle measure, betweenness, collinearity, and orientation) are preserved under this rotation. Is the rotation image of this regular pentagon congruent to the original regular pentagon?

Solution: In the adjacent diagram, regular pentagon *ABCDE* has been rotated 216° around vertex *D*, resulting in pentagon *A′B′C′D′E′*. Since a rotation is a composition of reflections over two intersecting lines, it preserves distance, angle measure, betweenness, collinearity, and orientation. Therefore, the two regular pentagons are congruent to each other.

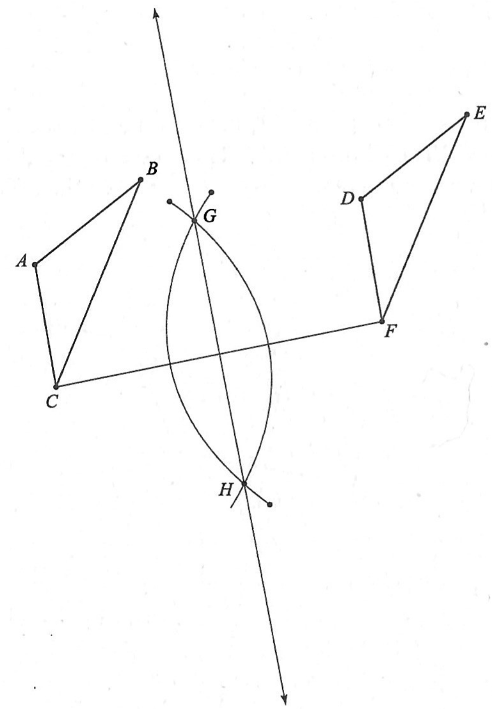


Example 2

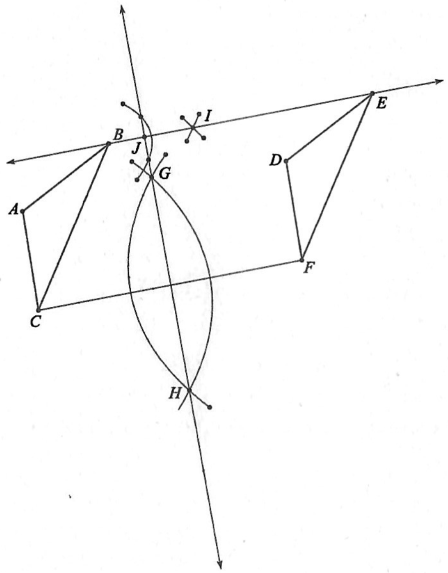
In the diagram below, $\triangle DEF$ is the translation image of $\triangle ABC$. Determine two parallel lines that are used to map $\triangle DEF$ to $\triangle ABC$. Why are these triangles congruent to each other?



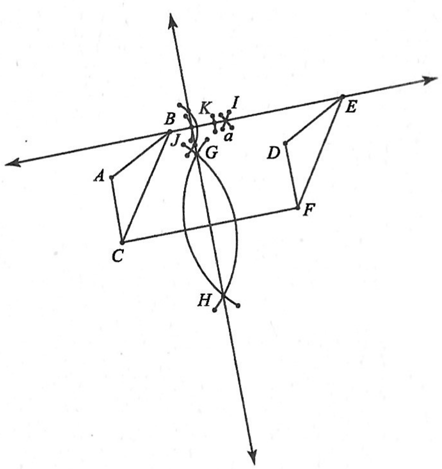
Solution: First construct the line segment $\overline{CF}$. Then construct its perpendicular bisector, $\overleftrightarrow{GH}$.



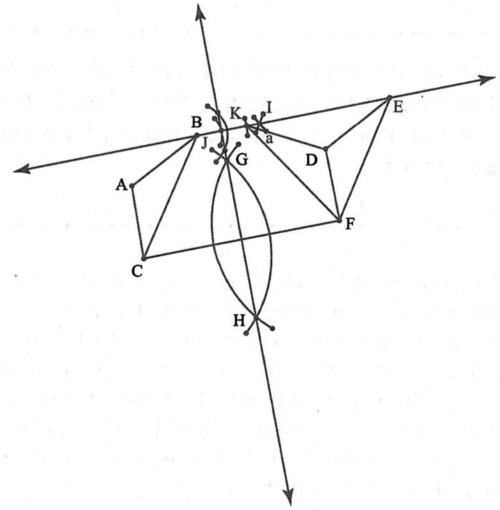
Now construct a line perpendicular to perpendicular bisector $\overleftrightarrow{GH}$ from point B. Call this line $\overleftrightarrow{BI}$.



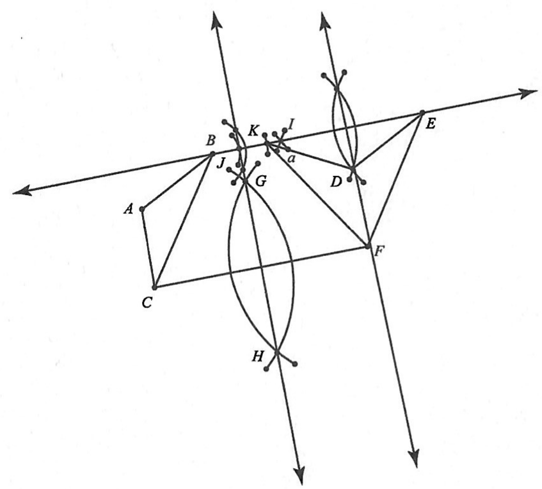
On this new perpendicular line $\overleftrightarrow{BI}$, measure the distance from point B to where it intersects $\overleftrightarrow{GH}$. Call this point J. Then measure the distance from B to J. Mark that distance on the other side of $\overleftrightarrow{GH}$ to find the reflection image of point B at point K.



Finally, draw $\overline{KD}$ and $\overline{KF}$.



$r_{\overleftrightarrow{GH}}(\triangle ABC) = \triangle DKF$. Now construct the perpendicular bisector of $\overline{KE}$.



In a plane, two lines perpendicular to the same line are parallel to each other. Therefore, $\overleftrightarrow{GH}$ is parallel to $\overleftrightarrow{DF}$. In fact, the composition of reflections over these lines creates the desired translation: $r_{\overleftrightarrow{GH}}(\triangle DKF) = \triangle DEF$. Since reflections (or a composition of reflections) preserves distance and angle measure, $\triangle ABC \cong \triangle DEF$.

G-CO.7

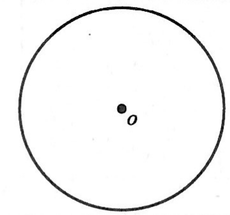
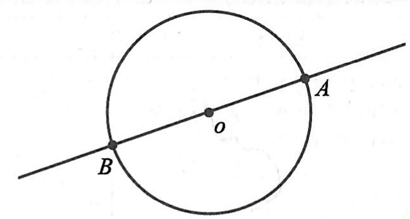
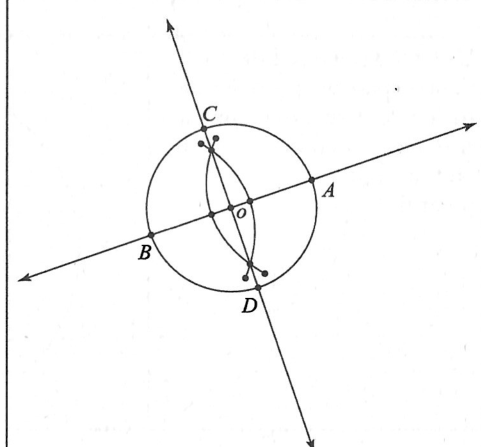
Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

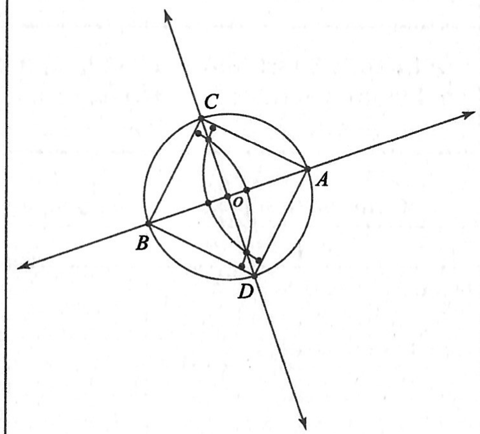
Two figures are congruent if a rigid motion transforms one of the figures to the other. Remember that all of the rigid motions are either line reflections or composites of line reflections. Also remember that line reflections preserve distance and angle measure. Therefore, triangles that are transformed through rigid motions have pairs of sides that are of equal length and are therefore congruent. These triangles also have pairs of angles of equal measure and are also therefore congruent. In that way, all six corresponding parts of congruent triangles must be congruent.

G-CO.13

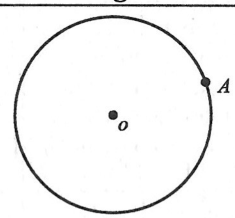
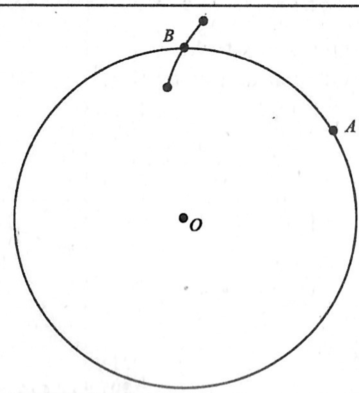
Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

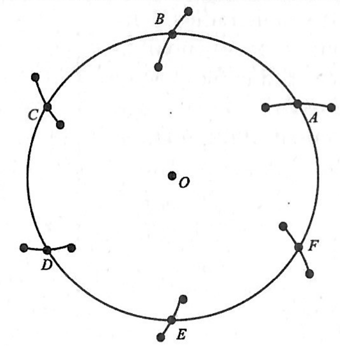
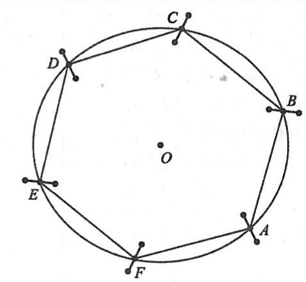
Construction 9—Construct a Square Inscribed in a Circle

Step	Diagram
Open a compass to a convenient radius, and draw a circle from point O.	
Locate a point anywhere on the circumference of this circle, and name it point A. Through points A and O, draw a line. Label the point on the circle on the other side of this line from point O as B.	
Construct a perpendicular line to $\overline{AB}$ at point O. Label the points on that perpendicular that lie on the circle as D and E.	

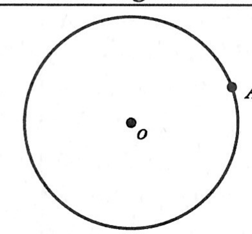
Step	Diagram
Draw line segments $\overline{AD}$, $\overline{BD}$, $\overline{BC}$, and $\overline{AC}$. $ADBC$ is the desired inscribed square.	

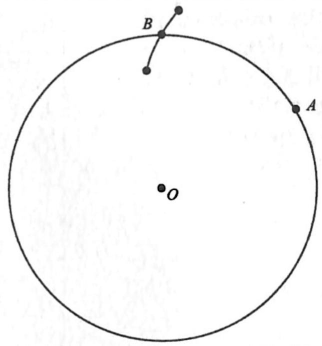
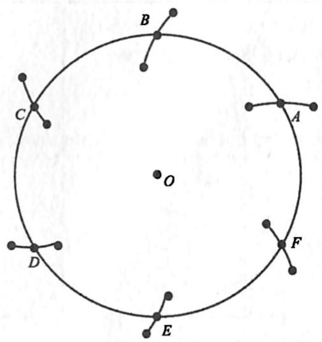
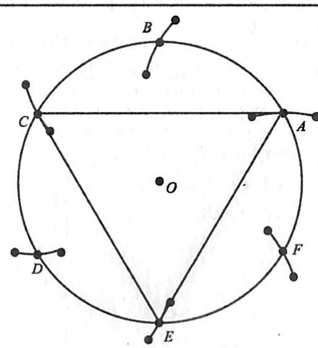
Construction 10—Construct a Hexagon Inscribed in a Circle

Step	Diagram
Open a compass to a convenient radius, and draw a circle from point O.	
Using the same radius, put the compass on any point on the circle. Label that point as A. Then mark off that length on the circle, and call that point B.	

Step	Diagram
Move the compass to point B. Mark off the same length, and call that point C. Keep moving the compass and labeling the points as D, E, and F until you reach point A.	
Connect the six points A, B, C, D, E, and F to form the hexagon.	

Construction 11—Construct an Equilateral Triangle Inscribed in a Circle

Step	Diagram
Open a compass to a convenient radius, and draw a circle from point O.	

Step	Diagram
Using the same radius, put the compass on any point on the circle. Label that point as *A*. Then mark off that length on the circle, and call that point *B*.	
Move the compass to point *B*. Mark off the same length, and call that point *C*. Keep moving the compass and labeling the points *C*, *D*, *E*, and *F* until you reach point *A*.	
Connect the three points *A*, *C*, and *E* to form the equilateral triangle. (Note: You can instead connect points *B*, *D*, and *F* to form an equilateral triangle.)	

G-SRT.1a

Verify experimentally the properties of dilations given by a center and a scale factor. Dilation takes a line not passing through the center of the dilation to a parallel line and leaves a line passing through the center unchanged.

Example 1

In the diagram below, $\overline{DE}$ is dilation image of $\overline{BC}$ through point *A*. The coordinates of the labeled points are *A*(1,2), *B*(2,5), *C*(5,3), *D*(4,11), and *E*(13,5). Using the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, determine the lengths of segments $\overline{AB}$, $\overline{AD}$, $\overline{AC}$, and $\overline{AE}$.

$$d_{AB} = \sqrt{(2-1)^2 + (5-2)^2} = \sqrt{1^2 + 3^2} = \sqrt{1+9} = \sqrt{10}$$

$$d_{AD} = \sqrt{(4-1)^2 + (11-2)^2} = \sqrt{3^2 + 9^2} = \sqrt{9+81} = \sqrt{90} = 3\sqrt{10}$$

$$d_{AC} = \sqrt{(5-1)^2 + (3-2)^2} = \sqrt{4^2 + 1^2} = \sqrt{16+1} = \sqrt{17}$$

$$d_{AE} = \sqrt{(13-1)^2 + (5-2)^2} = \sqrt{12^2 + 3^2} = \sqrt{144+9} = \sqrt{153} = 3\sqrt{17}$$

Clearly, the dilation through point *A* is of scale factor 3. Now find the lengths of segments $\overline{BC}$ and $\overline{DE}$.

$$d_{BC} = \sqrt{(5-2)^2 + (3-5)^2} = \sqrt{3^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$$

$$d_{DE} = \sqrt{(13-4)^2 + (5-11)^2} = \sqrt{9^2 + (-6)^2} = \sqrt{81+36} = \sqrt{117} = 3\sqrt{13}$$

So the scale factor of 3 also applies to $\overline{BC}$ and its dilation image $\overline{DE}$. Now let's examine the slopes of $\overline{BC}$ and its image $\overline{DE}$. The slope formula is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

lar prisms next to this upper rectangular solid also have the same width, w_1, with triangular base b_1 and height h_2. Each of these right triangular prisms has volumes equal to $\frac{1}{2}b_1h_2w_1$. Finally, the top triangular prism has a base of length l_2 and height of h_3 with that same length of w_1. Its volume is $\frac{1}{2}l_2h_3w_1$.

So the total volume of the barn is $l_1w_1h_1 + l_2w_1h_2 + b_1h_2w_1 + \frac{1}{2}l_2h_3w_1$.

G-MG.2

Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile or BTUs per cubic foot).

Density is equal to mass divided by volume. Use this formula to help solve problems regarding density.

Example 1
An average classroom at the Pinemeadows School holds 32 students. The classroom measures 25 feet wide, 28 feet long, and 9 feet high. What is the population density for an average classroom at the Pinemeadows School?

Solution: The volume of the average classroom is $25 \times 28 \times 9 = 6{,}300$ cubic feet. Since $d = \frac{m}{v}$, there are $\frac{32 \text{ students}}{6{,}300 \text{ ft}^3} \approx 0.005$ students per cubic foot.

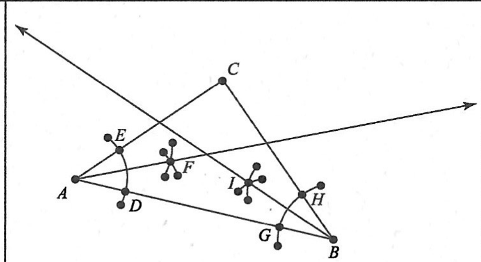
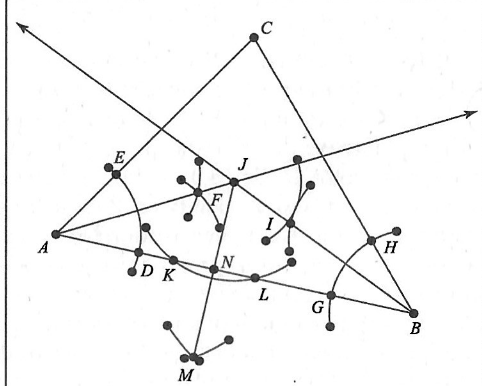
Example 2
One well-insulated home requires approximately 36,000 BTUs to heat the house properly. If the house has a floor space of approximately 1,500 square feet, how many BTUs are needed per square foot to heat the home?

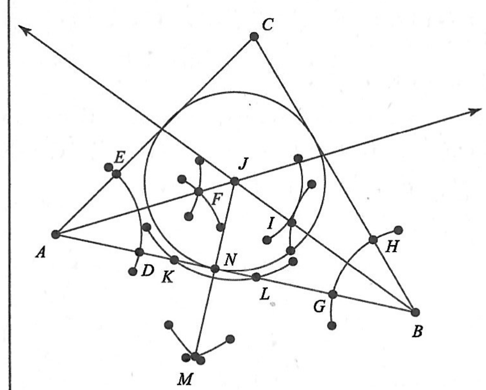
Solution: The number of BTUs required $= \frac{36{,}000 \text{ BTUs}}{1{,}500 \text{ square feet}} = 24$ BTUs per square foot.

G-C.3

Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

To construct a circle inscribed in a triangle, it is necessary to know that the point of concurrency of the three angle bisectors of a triangle is the center of the inscribed angle (called the incenter of the triangle). Therefore, bisecting two of the three angles of the triangle will find the center. Then use the length of a line segment from the incenter to a side as the radius and construct the incircle.

Step	Diagram
Bisect two of the three angles of the triangle.	
Construct a perpendicular line from the point of intersection, J, (this point is called the incenter of the triangle) to a side of the triangle. Name the point where this perpendicular meets side $\overline{AB}$ of the triangle as point N.	

Step	Diagram
Use the length of this perpendicular segment $\overline{JN}$ as the radius of a circle. Draw a circle from the incenter, point J.	

G-C.5

Use similarity to derive the fact that the length of the arc intercepted by an angle is proportional to the radius. Define the radian measure of the angle as the constant of proportionality. Derive the formula for the area of a sector.

In a circle whose radius is equal to 1 (sometimes referred to as a unit circle), there are 360° around the center. The circumference of the circle also contains 360°. Therefore, the measure of a central angle is equal to the measure of its intercepted arc. The same argument can be made if the measures are in radians because the number of radians around a point is 2π and the number of radians contained in the circumference of the circle is also 2π. If the radius of the circle is not equal to 1, it is important to recognize that the length of the intercepted arc will change. However, the degree and/or radian measure of this arc will still be measured by the proportion of the circle that contains a total of 360° or 2π radians. So the measure of a central angle of a circle (in degrees or in radian measure) is equal to the measure of its arc (in degrees or in radian measure). The relationships between the measure of a central angle and an inscribed angle, which is an angle formed by a tangent and a chord, between an angle formed by two chords intersecting inside a cir-